Regularized QSM in Seconds

Berkin Bilgic¹, Itthi Chatnuntawech², Audrey P. Fan², Elfar Adalsteinsson²,³

¹Martinos Center for Biomedical Imaging, Charlestown, MA, USA
²MIT, Cambridge, MA USA
³Harvard-MIT Health Sciences and Technology, Cambridge, MA USA
Declaration of Relevant Financial Interests or Relationships

Speaker Name: Berkin Bilgic

I have no relevant financial interest or relationship to disclose with regard to the subject matter of this presentation.
Quantitative Susceptibility Mapping (QSM)

- Quantitative Susceptibility Mapping (QSM) aims to quantify tissue magnetic susceptibility $\chi$
- Susceptibility correlates well with tissue iron concentration, especially in iron rich deep gray matter structures [1,2]

[1] Langkammer et al., Neuroimage 2012
Quantitative Susceptibility Mapping (QSM) aims to quantify tissue magnetic susceptibility $\chi$

Susceptibility correlates well with tissue iron concentration, especially in iron rich deep gray matter structures [1,2]

Susceptibility mapping requires the solution of an inverse problem,

$$F^H \cdot D \cdot F \cdot \chi = \phi$$

- $\chi$ is the unknown susceptibility
- $\phi$ is the unwrapped phase
- $D$ is the diagonal matrix
- $F$ is the DFT

[1] Langkammer et al., Neuroimage 2012
Quantitative Susceptibility Mapping (QSM) aims to quantify tissue magnetic susceptibility $\chi$.

Susceptibility correlates well with tissue iron concentration, especially in iron rich deep gray matter structures [1,2].

Susceptibility mapping requires the solution of an inverse problem,

$$F^H D F \chi = \phi$$

to be estimated measured

[1] Langkammer et al., Neuroimage 2012
Quantitative Susceptibility Mapping (QSM)

- Quantitative Susceptibility Mapping (QSM) aims to quantify tissue magnetic susceptibility $\chi$
- Susceptibility correlates well with tissue iron concentration, especially in iron rich deep gray matter structures
- Susceptibility mapping requires the solution of an inverse problem,

$$F^H D F \chi = \phi$$

$$D = \frac{1}{3} - \frac{k_z^2}{k^2}$$

Undersamples k-space on a conical surface
Regularized QSM

- Solution of inverse problem is facilitated by regularization that imposes prior knowledge [1]

\[
\chi = \arg\min_{\chi} \| \phi - F^H D F \chi \|_2^2 + \lambda \cdot \| G \chi \|_2^2
\]

- data consistency
- \( \ell_2 \) over gradients

Regularized QSM

- Solution of inverse problem is facilitated by regularization that imposes prior knowledge [1]

\[
\chi = \arg\min_\chi \| \phi - F^H D F \chi \|_2^2 + \lambda \cdot \| G \chi \|_2^2
\]

- data consistency
- \( \ell_2 \) over gradients

\[
G = \begin{bmatrix}
G_x \\
G_y \\
G_z
\end{bmatrix}
\]

gradient in 3D

Regularized QSM

- Solution of inverse problem is facilitated by regularization that imposes prior knowledge [1]

\[ \chi = \arg\min_{\chi} \| \phi - F^H DF \chi \|_2^2 + \lambda \cdot \| G \chi \|_2^2 \]

- Prior: underlying susceptibility map is smooth

**Regularized QSM**

- Solution of inverse problem is facilitated by regularization that imposes prior knowledge [1]

\[
\chi = \arg\min_{\chi} \| \phi - F^H D F \chi \|_2^2 + \lambda \cdot \| G \chi \|_2^2
\]

- Existing methods work iteratively [1,2], requiring ~30 minutes for a 3D volume → not feasible

- We address this with fast recon in ~1 second

Regularized QSM

- Solution of inverse problem is facilitated by regularization that imposes prior knowledge [1]

\[ \chi = \text{argmin}_\chi \| \phi - F^H DF \chi \|_2^2 + \lambda \cdot \| G \chi \|_2^2 \]

- Solution can be evaluated in closed-form

\[ \chi = (F^H D^2 F + \lambda \cdot G^H G)^{-1} F^H DF \phi \]

- The minimizer can be computed efficiently given that the matrix inversion is rapidly performed

Fast Regularized QSM

- Solution can be evaluated in closed-form

\[ \chi = (F^H D^2 F + \lambda \cdot G^H G)^{-1} F^H DF \phi \]
Fast Regularized QSM

- Solution can be evaluated in closed-form

\[ \chi = (F^H D^2 F + \lambda \cdot G^H G)^{-1} F^H D F \phi \]

- Gradient along x-axis can be represented in k-space by multiplication with a diagonal matrix \( E_x \)

\[ G_x = F^H E_x F \quad \text{where} \quad E_x(i, i) = 1 - e^{(-2\pi \sqrt{-1}k_x(i,i)/N_x)} \]
Fast Regularized QSM

- Solution can be evaluated in closed-form
  \[
  \chi = (F^H D^2 F + \lambda \cdot G^H G)^{-1} F^H D F \phi
  \]

- Gradient along x-axis can be represented in k-space by multiplication with a diagonal matrix \( E_x \)
  \[
  G_x = F^H E_x F \quad \text{where} \quad E_x(i, i) = 1 - e^{(-2\pi\sqrt{-1}k_x(i,i)/N_x)}
  \]

- \( E_x \) is simply the k-space representation of the difference operator \( \delta_x - \delta_{x-1} \)
Fast Regularized QSM

- Solution can be evaluated in closed-form
  \[ \chi = \left( F^H D^2 F + \lambda \cdot G^H G \right)^{-1} F^H D F \phi \]

- Gradient along x-axis can be represented in k-space by multiplication with a diagonal matrix \( E_x \)
  \[ G_x = F^H E_x F \quad \text{where} \quad E_x(i, i) = 1 - e^{(-2\pi\sqrt{-1} k_x(i,i)/N_x)} \]

- With this formulation, closed-form solution becomes
  \[ \chi = F^H D \left[ D^2 + \lambda \cdot (E_x^2 + E_y^2 + E_z^2) \right]^{-1} F \phi \]

  all matrices diagonal
Fast Regularized QSM

- Solution can be evaluated in closed-form

\[ \chi = (F^H D^2 F + \lambda \cdot G^H G)^{-1} F^H D F \phi \]

- Gradient along x-axis can be represented in k-space by multiplication with a diagonal matrix \( E_x \)

\[ G_x = F^H E_x F \]

where \( E_x(i, i) = 1 - e^{(-2\pi \sqrt{-1} k_x(i,i)/N_x)} \)

- With this formulation, closed-form solution becomes

\[ \chi = F^H D \left[ D^2 + \lambda \cdot (E_x^2 + E_y^2 + E_z^2) \right]^{-1} F \phi \]

- **Total cost:** Two FFTs and multiplication of diagonal matrices
Contributions

- Proposed closed-form method is 1000-times faster than iterative Conjugate Gradient solver in [1,2]

[1] de Rochefort et al., MRM 2010
Contributions

- Proposed closed-form method is 1000-times faster than iterative Conjugate Gradient solver in [1,2]

- Proposed method yields exact minimizer while iterative methods converge to it

[1] de Rochefort et al., MRM 2010
Contributions

- Proposed closed-form method is 1000-times faster than iterative Conjugate Gradient solver in [1,2]

- Proposed method yields exact minimizer while iterative methods converge to it

- Automatic selection of regularization parameter $\lambda$ is possible: Trace L-curve with closed-form method in a minute

[1] de Rochefort et al., MRM 2010
Contributions

- Proposed closed-form method is 1000-times faster than iterative Conjugate Gradient solver in [1,2]

- Proposed method yields exact minimizer while iterative methods converge to it

- Automatic selection of regularization parameter $\lambda$ is possible: Trace L-curve with closed-form method in a minute

- Combined with fast background removal methods like SHARP [3], enables real-time QSM

[1] de Rochefort et al., MRM 2010
Comparison of methods

- **Proposed method:**
  - Closed form QSM

- **Previous method:**
  - Iterative QSM with Conjugate Gradient [1,2] converges to closed-form solution

[1] de Rochefort et al., MRM 2010
Comparison of methods

- **Proposed method:**
  - Closed form QSM

- **Previous method:**
  - Iterative QSM with Conjugate Gradient [1,2] converges to closed-form solution
  - Initialize with Thresholded K-space Division map [3]
  - Terminate when change in susceptibility is less than 1%

[1] de Rochefort et al., MRM 2010
Regularized QSM Methods

- **Numerical Phantom**
  - Three compartments (gray, white, CSF) with constant $\chi$
  - Phase $\phi$ computed from true $\chi$, and Gaussian noise added
  - Regularization param $\lambda$ chosen to minimize RMSE in $\chi$ recon

[1] Liu et al., NMR in Biomed 2011
Regularized QSM Methods

- **Numerical Phantom**
  - Three compartments (gray, white, CSF) with constant $\chi$
  - Phase $\phi$ computed from true $\chi$, and Gaussian noise added
  - Regularization param $\lambda$ chosen to minimize RMSE in $\chi$ recon

- **In Vivo 3D SPGR**
  - Healthy subject at 1.5T with resolution $0.94 \times 0.94 \times 2.5\text{mm}^3$
  - Regularization parameter $\lambda$ chosen based on L-curve
  - Background phase removal with dipole fitting [1]

- Computations done on workstation with 32 CPU processors and 128 GB memory

[1] Liu et al., NMR in Biomed 2011
Numerical Phantom

Noisy phase $\phi$

- Error due to noise: 5.0% RMSE
- RMSE $= 0.01$ ppm

Closed-form QSM in 1.1 seconds

Closed-form QSM error relative to True $\chi$

- True $\chi$ known

MAGNIFIED 3 TIMES
 Numerical Phantom

- Noisy phase $\phi$

- Error due to noise: 5.0% RMSE

Closed-form QSM in 1.1 seconds

<table>
<thead>
<tr>
<th>QSM Method</th>
<th>Recon Time</th>
<th>Error relative to True $\chi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed Closed-Form</td>
<td><strong>1.1 seconds</strong></td>
<td>16.1 % RMSE</td>
</tr>
<tr>
<td>Conjugate Grad, 80 iters</td>
<td>33 minutes</td>
<td>16.8 % RMSE</td>
</tr>
</tbody>
</table>
In Vivo QSM

Tissue phase $\phi$

Closed-form QSM in 0.6 seconds

Closed-form and Iterative QSM difference: 0.6%

True $\chi$ not known

MAGNIFIED 100 TIMES
In Vivo QSM

Tissue phase $\phi$

Closed-form QSM in 0.6 seconds

<table>
<thead>
<tr>
<th>QSM Method</th>
<th>Recon Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed Closed-Form</td>
<td>0.6 seconds</td>
</tr>
<tr>
<td>Conjugate Gradient, 80 iters</td>
<td>18 minutes</td>
</tr>
</tbody>
</table>
Tracing the L-curve

- Computing \( \chi \) for 25 different values of \( \lambda \): 50 seconds
Tracing the L-curve

\[ \| G \chi \|_2 \]

\[ \| \phi - F^H D F \chi \|_2 \]

- Computing \( \chi \) for 25 different values of \( \lambda \): 50 seconds
- Find optimal \( \lambda \) by computing the curvature of L-curve

Largest curvature on L-curve: \( \lambda = 0.013 \)
Tracing the L-curve

\[ \| Gx \|_2 \]

\[ \| \phi - F^HDFx \|_2 \]

Under-regularized \( \lambda = 0.001 \)
Tracing the L-curve

\[ \| \mathbf{G} \chi \|_2 \]

Under-regularized \( \lambda = 0.001 \)

Optimally-regularized \( \lambda = 0.013 \)
Tracing the L-curve

\[ \|G\chi\|_2 \]

Under-regularized \( \lambda = 0.001 \)

Optimally-regularized \( \lambda = 0.013 \)

Over-regularized \( \lambda = 0.091 \)
Conclusion

- Proposed closed form recon for L2-regularized QSM
- 1000-times faster recon compared to Conjugate Gradient solver [1,2]
- Automatic selection for $\lambda$ feasible with L-curve in a minute
- Software Download:

  http://web.mit.edu/berkin/www/software.html

[1] de Rochefort et al., MRM 2010
Acknowledgments

- **Sponsors:**
  - MIT-CIMIT Medical Engineering Fellowship
  - Siemens Healthcare
  - Siemens-MIT Alliance

- **Grants:**
  - K99EB012107, U01MH093765,
  - R01EB006847, R01EB007942,
  - R01EB000790, P41RR14075