Fast Reconstruction for Regularized Quantitative Susceptibility Mapping

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Declaration of Financial Interests or Relationships

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I have no financial interests or relationships to disclose with regard to the subject matter of this presentation.
Quantitative Susceptibility Mapping (QSM)

- QSM estimates the underlying magnetic susceptibility that gives rise to subtle changes in the magnetic field.

- Estimation of the susceptibility map $\chi$ from the unwrapped phase $\phi$ involves solving an inverse problem\(^1\),

$$
\delta = F^{-1} D F \chi
$$

$\delta$: Measured estimate

$\phi$: Normalized field map

1 Marques JP et al., Concepts in Magn Res 2005
Quantitative Susceptibility Mapping (QSM)

- QSM estimates the underlying magnetic susceptibility that gives rise to subtle changes in the magnetic field.

- Estimation of the susceptibility map $\chi$ from the unwrapped phase $\phi$ involves solving an inverse problem,

$$\delta = F^{-1} D F \chi$$

- The inversion is made difficult by zeros in susceptibility kernel $D$

$$D = \frac{1}{3} - \frac{k_z^2}{k_x^2 + k_y^2 + k_z^2}$$
Quantitative Susceptibility Mapping (QSM)

- QSM estimates the underlying magnetic susceptibility that gives rise to subtle changes in the magnetic field.
- Estimation of the susceptibility map $\chi$ from the unwrapped phase $\phi$ involves solving an inverse problem,

$$\delta = F^{-1} D F \chi$$

- The inversion is made difficult by zeros in susceptibility kernel $D$.
- Undersampling is due to physics

Not in our control.
Regularized Susceptibility Inversion

- Regularized QSM imposes smoothness or sparsity constraints on the gradient of the susceptibility map.

- L2-regularization\(^1,^2\) (smoothness prior):

\[
\min \left\| F^{-1}DF\chi - \delta \right\|_2^2 + \beta \cdot \left\| MG\chi \right\|_2^2
\]

Data consistency

Regularizer

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\(^1\) De Rochefort L et al., MRM 2010  \(^2\) Liu T et al., MRM 2011
Regularized Susceptibility Inversion

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- L2-regularization\(^1,2\) (smoothness prior):

\[
\min \left\| F^{-1}DF \chi - \delta \right\|^2_2 + \beta \cdot \left\| MG \chi \right\|^2_2
\]

- \(G\) : Spatial gradient operator in 3D
- \(M\) : Binary mask derived from magnitude image, prevents smoothing across edges
- \(\beta\) : Determines the amount of smoothness

\(^1\) De Rochefort L et al., MRM 2010  \(^2\) Liu T et al., MRM 2011
Regularized Susceptibility Inversion

- Regularized QSM imposes smoothness or sparsity constraints on the gradient of the susceptibility map

- L2-regularization\(^1,2\) (smoothness prior):
  \[
  \min \left\| F^{-1}DF \chi - \delta \right\|_2^2 + \beta \cdot \left\| MG \chi \right\|_2^2
  \]

- L1-regularization\(^3,4\) (sparsity prior):
  \[
  \min \left\| F^{-1}DF \chi - \delta \right\|_2^2 + \alpha \cdot \left\| MG \chi \right\|_1
  \]

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1 De Rochefort L et al., MRM 2010  
2 Liu T et al., MRM 2011  
3 Liu J et al., NIMG 2012  
4 Wu B et al., MRM 2012
Regularized Susceptibility Inversion

- Regularized QSM imposes smoothness or sparsity constraints on the gradient of the susceptibility map
- L2-regularization\(^{1,2}\) (smoothness prior)
- L1-regularization\(^{3,4}\) (sparsity prior)
- Reported reconstruction times are in the range between 20 minutes\(^{2,3}\) to 2-3 hours\(^4\)
- We propose efficient solvers that are up to \(20 \times\) faster
- Facilitate online recon and clinical application of QSM

Matlab Software: martinos.org/~berkin

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1 De Rochefort L et al., MRM 2010  
2 Liu T et al., MRM 2011  
3 Liu J et al., NIMG 2012  
4 Wu B et al., MRM 2012
3D GRE 0.6 mm iso

L2 Regularized

Closed-form L2$^1$
Recon time: 0.9 sec

$^1$ Bilgic B et al., JMRI 2013
L2 Regularized QSM

\[
\min \left\| F^{-1} D F \chi - \delta \right\|^2_2 + \beta \cdot \left\| G \chi \right\|^2_2
\]

- Without magnitude weighting (\(M=\text{Identity}\)), we proposed a closed-form solution\(^1\)
- This relies on computing gradients in k-space rather than image-space:

\[
G = F^{-1} E F \quad E : \text{Diagonal}
\]

- With this trick, solution requires only two FFTs:

\[
\chi = F^{-1} \left( D^2 + \beta \cdot E^2 \right)^{-1} \cdot D F \delta
\]

\(^1\) Bilgic B et al., JMRI 2013

Matlab Software:
martinos.org/~berkin
L2 Regularized QSM

\[ \min \left\| F^{-1}DF \chi - \delta \right\|_2^2 + \beta \cdot \left\| G \chi \right\|_2^2 \]

• *Without* magnitude weighting (M=Identity), we proposed a closed-form solution

• This relies on computing gradients in k-space rather than image-space

• With this trick, solution requires only two FFTs

• Elegant improvements to closed-form L2:
  - Khabipova et al #602 and Schweser et al #605

Matlab Software:
martinos.org/~berkin

1 Bilgic B et al., JMRI 2013
L2 with Magn Weight

3D GRE 0.6 mm iso

Proposed
Recon time: 88 sec
L2 Regularization with Magnitude Weighting

\[
\min \left\| F^{-1}DF \chi - \delta \right\|_2^2 + \beta \cdot \left\| MG \chi \right\|_2^2
\]

- When magnitude weighting is included, optimizer is given by the solution of:

\[
(D^2 + \beta \cdot E^H [FMF^{-1}] E) F \chi = DF \delta
\]

This term cancels if \( M = I \)

Matlab Software: martinos.org/~berkin
L2 Regularization with Magnitude Weighting

\[
\min \left\| F^{-1}DF \chi - \delta \right\|^2_2 + \beta \cdot \left\| MG \chi \right\|^2_2
\]

- When magnitude weighting is included, optimizer is given by the solution of:
  \[
  (D^2 + \beta \cdot E^H F MF^{-1} E) F \chi = DF \delta
  \]
- Large linear system, solve iteratively with Conjugate Gradient (CG)
- Proposal: Use closed-form solution to speed-up convergence of Conjugate Gradient

Matlab Software: martinos.org/~berkin
L2 Regularization with Magnitude Weighting

\[
\min \left\| F^{-1} DF \chi - \delta \right\|_2^2 + \beta \cdot \left\| MG \chi \right\|_2^2
\]

- When magnitude weighting is included, optimizer is given by the solution of:

\[
(D^2 + \beta \cdot E^H F M F^{-1} E) F \chi = DF \delta
\]

\[
\text{call A}
\]
min \left\| F^{-1}DF \chi - \delta \right\|^2_2 + \beta \cdot \left\| MG \chi \right\|^2_2

- When magnitude weighting is included, optimizer is given by the solution of:

\[ A F \chi = DF \delta \]
L2 Regularization with Magnitude Weighting

\[
\min \left\| F^{-1}DF \chi - \delta \right\|^2_2 + \beta \cdot \left\| MG \chi \right\|^2_2
\]

- When magnitude weighting is included, optimizer is given by the solution of:
  \[
  AF \chi - DF \delta = 0
  \]

- The convergence speed of CG depends on the condition number of \( A \)

- Bring \( A \) closer to being identity using a preconditioner

Matlab Software: martinos.org/~berkin
L2 Regularization with Magnitude Weighting

\[
\min \left\| F^{-1}DF \chi - \delta \right\|_2^2 + \beta \cdot \left\| MG \chi \right\|_2^2
\]

• When magnitude weighting is included, optimizer is given by the solution of:

\[
(D^2 + \beta \cdot E^2)^{-1} \cdot (AF \chi - DF \delta) = 0
\]

\[
\text{closed-form}
\]

• Approximation:

\[
(D^2 + \beta \cdot E^2)^{-1} \approx A^{-1}
\]

• Preconditioned CG allows fast L2-regularization with Magnitude Weighting

Matlab Software:
martinos.org/~berkin
3D GRE 0.6 mm iso

L1 Regularized  L1 with Magn. Weight

Closed-form Recon time: 0.9 sec  Proposed Recon time: 88 sec
Proposed Recon time: 60 sec  Proposed Recon time: 275 sec
L1 Regularized QSM

\[ \min \left\| F^{-1}DF \chi - \delta \right\|_2^2 + \alpha \cdot \left\| MG \chi \right\|_1 \]

- L1-regularization has no closed-form solution, need to use expensive iterative methods
- Proposal: separate L1-regularization into simpler L2-regularization and soft thresholding problems
- Employ closed-form solution to solve L2-problem

Matlab Software: martinos.org/~berkin
L1 Regularized QSM

\[
\min \left\| F^{-1}DF \chi - \delta \right\|_2^2 + \alpha \cdot \left\| MG \chi \right\|_1
\]
L1 Regularized QSM

\[
\min \left\| F^{-1}DF \chi - \delta \right\|_2^2 + \alpha \cdot \| y \|_1
\]

auxiliary variable

Matlab Software:
martinos.org/~berkin
L1 Regularized QSM

\[ \min \left\| F^{-1}DF \chi - \delta \right\|^2_2 + \alpha \cdot \|y\|_1 \]

\[ \text{st } y = MG \chi \]

- Variable-splitting\textsuperscript{1,2} separates into simpler problems
  1) **L2-regularized:**
  \[ \chi_{t+1} = \arg\min_{\chi} \left\| F^{-1}DF \chi - \delta \right\|^2_2 + \mu \left\| y_t - MG \chi \right\|^2_2 \]
  2) **Soft thresholding:**
  \[ y_{t+1} = \arg\min_{y} \alpha \cdot \|y\|_1 + \mu \left\| y - MG \chi_{t+1} \right\|^2_2 \]

\textsuperscript{1} Goldstein T \textit{et al}, SIAM 2009  \hspace{1cm} \textsuperscript{2} Chen Z \textit{et al}, J Comp Assist Tomogr 2012

Matlab Software: \texttt{martinos.org/~berkin}
L1 Regularized QSM

\[
\min \left\| F^{-1}DF \chi - \delta \right\|_2^2 + \alpha \cdot \| y \|_1 \\
\text{st} \quad y = MG\chi
\]

- Variable-splitting\(^1,\)\(^2\) separates into simpler problems

1) L2-regularized:

\[
\chi_{t+1} = \arg\min_{\chi} \left\| F^{-1}DF \chi - \delta \right\|_2^2 + \mu \left\| y_t - MG\chi \right\|_2^2
\]

\[
\mu \text{ affects convergence, not final solution}\(^1\)
\]

Matlab Software: 
martinos.org/~berkin

\(^1\) Goldstein T et al, SIAM 2009
\(^2\) Chen Z et al, J Comp Assist Tomogr 2012
L1 Regularized QSM

\[ \min \left\| F^{-1}DF \chi - \delta \right\|^2_2 + \alpha \cdot \|y\|_1 \]

\[ \text{st } y = MG \chi \]

- Variable-splitting\(^1,^2\) separates into simpler problems

1) L2-regularized:

\[ \chi_{t+1} = \arg\min_\chi \left\| F^{-1}DF \chi - \delta \right\|^2_2 + \mu \left\| y_t - MG \chi \right\|^2_2 \]

- Very similar to L2-regularized QSM:

\[ \min \left\| F^{-1}DF \chi - \delta \right\|^2_2 + \beta \left\| MG \chi \right\|^2_2 \]

Use Preconditioned Conjugate Gradient

\(^1\) Goldstein T et al, SIAM 2009  \hspace{1cm} ^2\) Chen Z et al, J Comp Assist Tomogr 2012

Matlab Software: martinos.org/~berkin
L1 Regularized QSM

\[
\min \left \| F^{-1}DF \chi - \delta \right \|^2_2 + \alpha \cdot \|y\|_1 \\
\text{st } y = MG \chi
\]

• Variable-splitting\(^1,^2\) separates into simpler problems

2) Soft thresholding:

\[
y_{t+1} = \arg \min_y \alpha \cdot \|y\|_1 + \mu \|y - MG \chi_{t+1}\|^2_2
\]

• Closed-form solution with point-wise operations:

\[
y_{t+1} = \max \left(\left|MG \chi_{t+1}\right| - \frac{\alpha}{2\mu}, 0\right) \cdot \text{sign}(MG \chi_{t+1})
\]

\(^{1}\) Goldstein T et al, SIAM 2009  \(^{2}\) Chen Z et al, J Comp Assist Tomogr 2012

Matlab Software: martinos.org/~berkin
L1 Regularized QSM

\[
\min \left\| F^{-1}DF \chi - \delta \right\|_2^2 + \alpha \cdot \| y \|_1
\]

\[
st \ y = MG \chi
\]

- Variable-splitting\(^1,\(^2\) separates into simpler problems

1) L2-regularized:

Use Preconditioned CG

Iterate until converged

2) Soft thresholding:

\[
y_{t+1} = \max \left( \left| MG \chi_{t+1} \right| - \frac{\alpha}{2 \mu}, 0 \right) \cdot \text{sign}(MG \chi_{t+1})
\]

\(^1\) Goldstein T et al, SIAM 2009
\(^2\) Chen Z et al, J Comp Assist Tomogr 2012

Matlab Software: martinos.org/~berkin
Data Acquisition

- **High-resolution 3D GRE**
  - 0.6 mm isotropic at 3T
  - TR / TE = 26 / 8.1 ms
  - $R_{\text{inplane}} = 2$, Partial Fourier = 3/4
  - $T_{\text{acq}} = 16$ min

- **Simultaneous Multi-Slice EPI**
  - 2 mm isotropic at 7T
  - TR/TE$_1$/.../TE$_4 = 2040/15/74$ ms
  - $R_{\text{inplane}} = 3$, Multi-Band = 3
  - $T_{\text{acq}} = 2$ sec

Matlab Software: martinos.org/~berkin
Phase Processing  3D GRE 0.6 mm iso

Wrapped Phase

Unwrapped Phase

Tissue Phase

Laplacian unwrapping\textsuperscript{1}: 6 seconds

SHARP filtering\textsuperscript{2}: 7 seconds

\textsuperscript{1} Li W et al, Neuroimage 2012
\textsuperscript{2} Schweser F et al, Neuroimage 2011

Matlab Software: martinos.org/~berkin
Regularized QSM

Closed-form L2
Recon time: 0.9 sec

Proposed L2 with Magn Weight
Recon time: 88 sec

Proposed L1
Recon time: 60 sec

Vessels more visible with Magn Weight
Recon time: 275 sec

Max Intensity Projections over 3mm Slab

Matlab Software: martinos.org/~berkin
Comparing L1-Regularized QSM Methods

### Proposed L1
- Recon time: 60 sec, 13 iterations

### Nonlinear Conjugate Gradient L1
- Recon time: 1350 sec, 50 iterations

**20× speed-up**

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1 Lustig M et al, MRM 2007
2 Bilgic B et al, Neuroimage 2012

Matlab Software: martinos.org/~berkin
Maximum Intensity Projections

Closed-form L2, Recon time: 0.9 sec

Proposed L1, Recon time: 60 sec

3D GRE 0.6 mm iso

Proposed L2 with Magn Weight, Recon time: 88 sec

Proposed L1 with Magn Weight, Recon time: 275 sec

Matlab Software: martinos.org/~berkin
K-Space View in log scale

3D GRE 0.6 mm iso

Closed-form L2

Proposed L2 with Magn Weight

Proposed L1

Proposed L1 with Magn Weight

Magic angle compensated with Magn Weight

Matlab Software: martinos.org/~berkin
SMS EPI, 2 mm iso @ 7T
\(R_{\text{inplane}} = 3\), Multi-Band = 3
2 second acquisition

Fast recon may facilitate functional QSM\(^1,2\)

1 Balla D et al, ISMRM 2012
2 Bianciardi M et al, HBM 2013
Fast Regularized QSM: Conclusion

• Proposed rapid L1- and L2-regularized QSM algorithms that yield up to $20\times$ speed-up

• Extended these to admit magnitude weighted regularization for improved reconstruction

• When combined with fast phase processing methods, these may facilitate online recon and clinical QSM

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